Hybrid-Trefftz Finite Elements for Non-Homogeneous Wave Propagation Problems

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The idea

Ritz and Trefftz finite element methods

Comparison: approximation strategies of Ritz vs Trefftz

Ritz Method (1909):

- ✓ use shape functions that satisfy exactly **the boundary conditions** (Dirichlet and interior)
- ✓ combine them to *more or less* satisfy the domain equations
- ✓ basis of the conforming finite element

Trefftz Method (1926):

- ✓ use shape functions that satisfy exactly the **domain equations**
- ✓ combine them to *more or less* satisfy the boundary conditions
- ✓ basis of the boundary/meshless/Trefftz methods

Ritz and Trefftz finite element methods

Comparison: the approximation functions, the mesh and the user's perspective

	Shape functions
Ritz FEM	• **simple & general, can be used for any physical problem** • bear no physical meaning • determined by the **nodes** of the mesh
Trefftz FEM	• problem-dependent • only suitable for the problem they are computing • bear physical meaning • independent of the nodes

Ritz and Trefftz finite element methods

Comparison: the approximation functions, the mesh and the user's perspective

	Shape functions	Mesh
Ritz FEM	• **simple & general, can be used for any physical problem** • bear no physical meaning • determined by the **nodes** of the mesh	• needs to be conforming • refined meshes • sensitive to distortion • sensitive to high frequencies
Trefftz FEM	• problem-dependent • only suitable to the problem they are computing • bear physical meaning • independent of the nodes	• could be missing altogether… • coarse meshes • insensitive to distortion • **insensitive to high frequencies**

Ritz and Trefftz finite element methods

Comparison: the approximation functions, the mesh and the user's perspective

	Shape functions	Mesh	User's perspective
Ritz FEM	• **simple & general, can be used for any physical problem** • bear no physical meaning • determined by the **nodes** of the mesh	• needs to be conforming • refined meshes • sensitive to distortion • sensitive to high frequencies	• displacement solutions are typically better than stress solutions • commercially available • easy to use
Trefftz FEM	• problem-dependent • only adequate to the problem they are computing • bear physical meaning • independent of the nodes	• could be missing altogether… • coarse meshes • insensitive to distortion • **insensitive to high frequencies**	• displacement and stress solutions are balanced • commercially unavailable • harder to use

The formulation

Description of the problem

Transient wave propagation problem

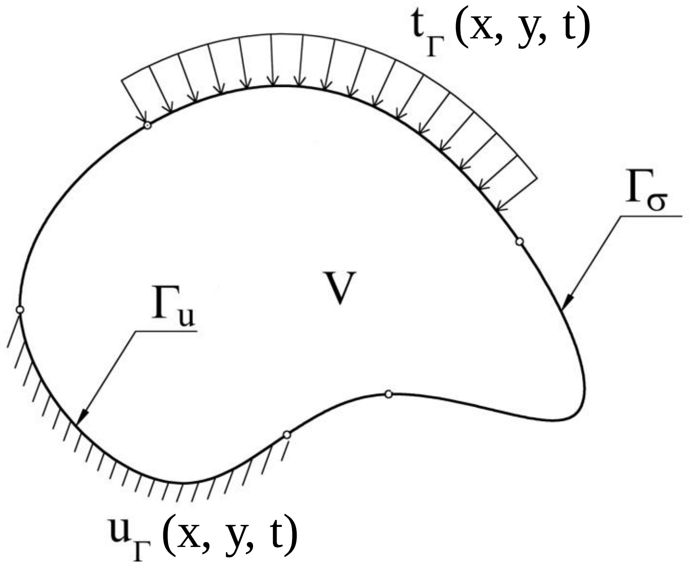
Find the displacement field $\boldsymbol{u}(\boldsymbol{x})$ and the stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ that satisfy the PDE.

Governing equations in the time domain

Domain	Navier	$\mathbf{D\,k\,D} * \mathbf{u}\,(\mathrm{x}, \mathrm{y}, \mathrm{t}) = \boldsymbol{\rho}\,\ddot{\mathbf{u}}\,(\mathrm{x}, \mathrm{y}, \mathrm{t})$

Boundaries	Dirichlet	$\mathbf{u}\,(\mathrm{x}, \mathrm{y}, \mathrm{t}) = \mathbf{u}_{\Gamma}\,(\mathrm{x}, \mathrm{y}, \mathrm{t})$
	Neumann	$\mathbf{n} \cdot \boldsymbol{\sigma}\,(\mathrm{x}, y, t) = \mathbf{t}_{\Gamma}\,(\mathrm{x}, y, t)$

Initial Conditions	$\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{0}) = \mathbf{u_0}(\mathbf{x}, \mathbf{y})$
	$\dot{\mathbf{u}}(\mathbf{x}, \mathbf{y}, \mathbf{0}) = \mathbf{v_0}(\mathbf{x}, \mathbf{y})$
	$\ddot{\mathbf{u}}(\mathbf{x}, \mathbf{y}, \mathbf{0}) = \mathbf{a_0}(\mathbf{x}, \mathbf{y})$



Description of the problem

Transient wave propagation problem

Governing equation in the time domain
Navier

$$\mathbf{D}\,\mathbf{k}\,\mathbf{D} * \; \mathbf{u}\,(x, y, t) = \boldsymbol{\rho}\,\ddot{\mathbf{u}}\,(x, y, t)$$

⇓

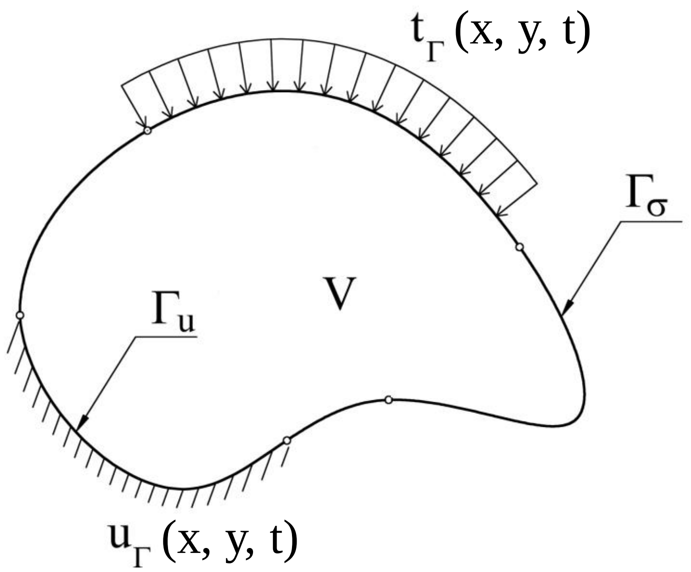
Discretization in time

Outcome: a series of spectral problems in space

⇓

Discretization in space

Solution of the problem discretized in time



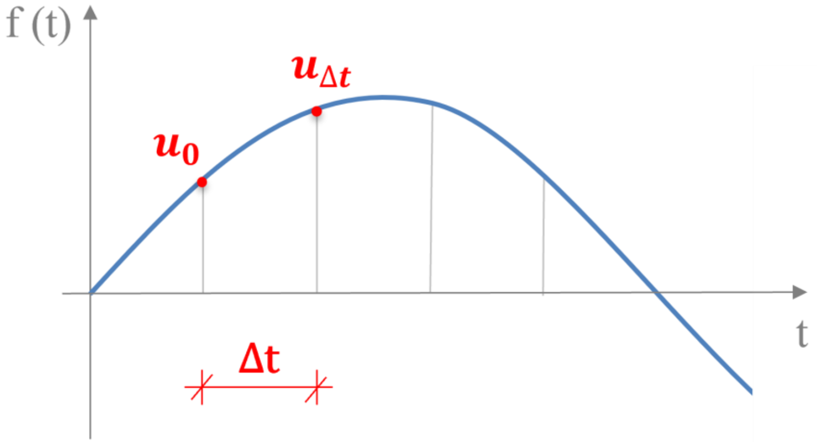
Solution overview

Discretization in time with the Newmark Method

The Newmark Method (second-order transient problems)

- Time stepping method
- Based on Taylor series

$$\omega^2 = \frac{1}{\beta \cdot \Delta t^2}$$



Known values	u_0
	v_0
	a_0

Unknown values	$u_{\Delta t}$
	$v_{\Delta t}$
	$a_{\Delta t}$

Series of pseudo spectral problems

Non-homogeneous equations

Solution overview

Discretization in space – displacement field approximation

$$\omega^2 = \frac{1}{\beta \cdot \Delta t^2}$$

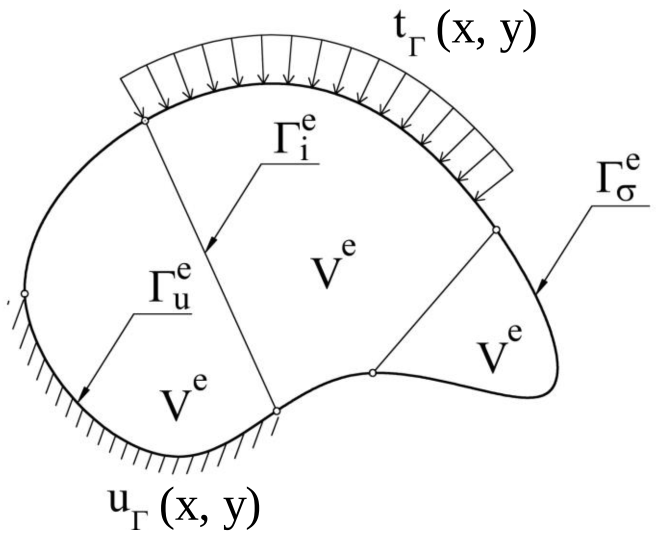
Navier equation for the spectral problem at the end of the time step

$$\mathbf{D\,k\,D} * \mathbf{u} + \omega^2 \boldsymbol{\rho}\, \mathbf{u} = \omega^2 \boldsymbol{\rho}\, \bar{\mathbf{u}}_0$$

General solution –approximation of the displacements

$$\mathbf{u} = \boldsymbol{\Psi}_c\ \mathbf{X}_c + \boldsymbol{\Psi}_p\ \mathbf{X}_p$$

Complementary solution basis — Particular solution basis



Must satisfy the homogeneous equation

$$\mathbf{D\,k\,D} * \boldsymbol{\Psi}_c + \omega^2 \boldsymbol{\rho}\ \boldsymbol{\Psi}_c = 0$$

Trefftz basis

Must satisfy the non-homogeneous equation

$$\mathbf{D\,k\,D} * \boldsymbol{\Psi}_p + \omega^2 \boldsymbol{\rho}\ \boldsymbol{\Psi}_p = \omega^2 \boldsymbol{\rho}\, \bar{\mathbf{u}}_0$$

? ?

→ **Dual Reciprocity Method**

Solution overview

Approximation of the particular solution - Conventional Dual Reciprocity Method

Approximation bases

Radial basis functions:

$$F_p^n(r) = r^{2n} \cdot \log(r)$$

Solution overview

Approximation of the particular solution - Conventional Dual Reciprocity Method

Approximation bases

Radial basis functions:

$$F_p^n(r) = r^{2n} \cdot \log(r)$$

$$U_p^n(r) = -\frac{[(2n)!!]^2}{\omega^{2n+2}} K_0(\omega r) + \sum_{k=1}^{n+1} c_k r^{2k-2} \log(r) + \sum_{k=1}^{n+1} d_k r^{2k-2}$$

$$c_k = -\frac{[(2n)!!]^2}{[(2k-2)!!]^2} \omega^{2k-2n-4}; \qquad d_k = c_k \sum_{j=k}^{n} \left(\frac{1}{j}\right)$$

Any other way?

Solution overview

Approximation of the particular solution

New Dual Reciprocity Method

$$\mathbf{D\,k\,D} * \boldsymbol{\Psi}_p\ \mathbf{X}_p + \omega^2\,\boldsymbol{\rho}\,\boldsymbol{\Psi}_p\ \mathbf{X}_p = \omega^2\,\boldsymbol{\rho}\,\bar{\mathbf{u}}_0$$

Non-homogeneous equation

$$\bar{\mathbf{u}}_0 = \boldsymbol{\Psi}_0\ \mathbf{X}_p$$

Approximation of the source term

? ? ?

$$\mathbf{D\,k\,D} * \boldsymbol{\Psi}_p + \omega^2\,\boldsymbol{\rho}\,\boldsymbol{\Psi}_p = \omega^2\,\boldsymbol{\rho}\,\boldsymbol{\Psi}_0$$

New Dual Reciprocity Method

- Define $\boldsymbol{\Psi}_p$ to obtain $\boldsymbol{\Psi}_0$

Assuming that $\boldsymbol{\Psi}_p$ satisfy a Trefftz condition:

$$\mathbf{D\,k\,D} * \boldsymbol{\Psi}_p + \lambda^2\,\boldsymbol{\rho}\,\boldsymbol{\Psi}_p = 0$$

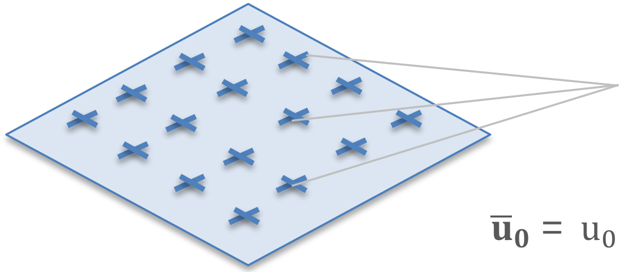
$$\Leftrightarrow \mathbf{D\,k\,D} * \boldsymbol{\Psi}_p = -\lambda^2\,\boldsymbol{\rho}\,\boldsymbol{\Psi}_p$$

$$\boldsymbol{\Psi}_0 = \frac{\omega^2 - \lambda^2}{\omega^2}\,\boldsymbol{\Psi}_p \qquad (\lambda \neq \omega)$$

Source term and particular solution bases are equivalent!

Solution overview

Implementation of the New Dual Reciprocity Method

1	Choice of one or more λ pseudo-frequencies	$\lambda = \{1, 2, 3\}$
2	Construction of the particular solution basis $\mathbf{\Psi_p}$ (Trefftz with λ)	$\mathbf{\Psi_p} = \begin{pmatrix} \beta_p^{-1}(J_{n-1} - J_{n+1}) & i\beta_s^{-1}(J_{n-1} + J_{n+1}) \\ i\beta_p^{-1}(J_{n-1} + J_{n+1}) & \beta_s^{-1}(J_{n+1} - J_{n-1}) \end{pmatrix}$ Trefftz with λ
3	Obtaining the source term basis $\mathbf{\Psi_0}$	$\mathbf{\Psi_0} = \frac{\omega^2 - \lambda^2}{\omega^2} \mathbf{\Psi_p}$
4	Collocation of $\mathbf{\bar{u}_0}$ in Gauss points	 Gauss collocation points where $\mathbf{\bar{u}_0}$ is known $\mathbf{\bar{u}_0} = u_0 + \Delta t \cdot v_0 + (1/2 - \beta) \cdot \Delta t^{2} \cdot a_0$
5	Solving the system to obtain the $\mathbf{X_p}$ weights	$\mathbf{X_p} = \mathbf{\bar{u}}_0 \cdot \mathbf{\Psi}_0^{-1}$
6	Obtaining the particular solution $\mathbf{u_p}$	$\mathbf{u_p} = \mathbf{\Psi_p}\ \mathbf{X_p}$

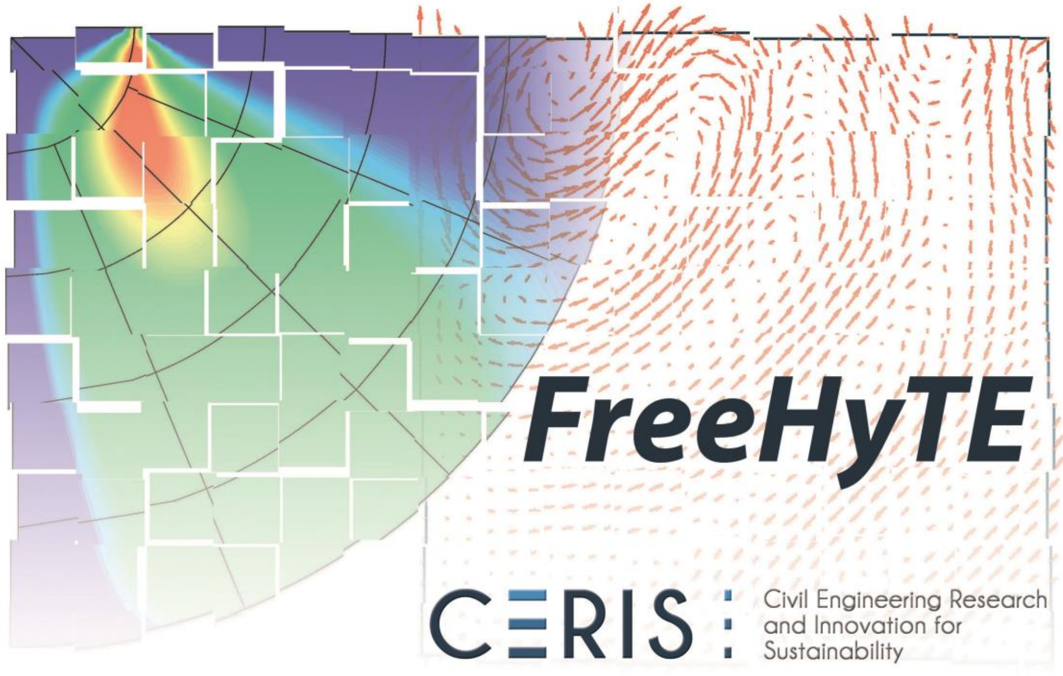
Solution overview

New Dual Reciprocity Method

New Dual Reciprocity Method approach	
Advantages	• **Simple expressions** for the particular solution approximation • Regular approximation functions, **no points of singularity** • Flexible definition – choices of λ are free • **Simple implementation** because the shape functions for the complementary and for the particular solution are similar • All coefficients are defined by boundary integrals
Drawback	• The collocation system may be ill-conditioned

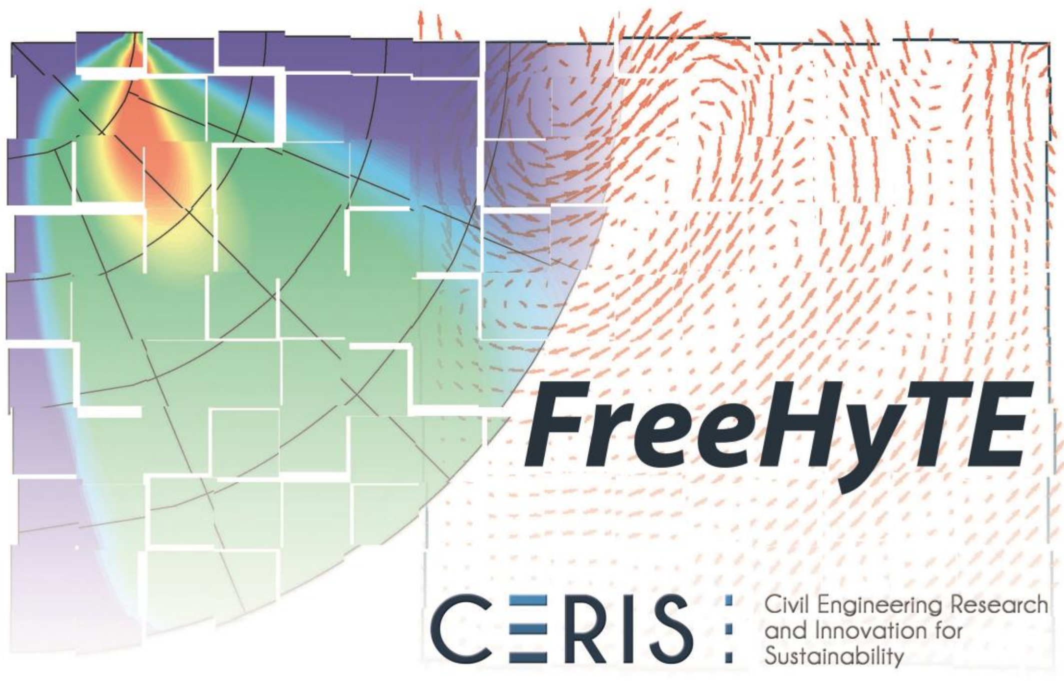
Moldovan ID, Radu L - Trefftz-based Dual Reciprocity Method for hyperbolic boundary value problems, International Journal for Numerical Methods in Engineering, 106(13), 1043-1070, 2016

The implementation



What is FreeHyTE?

- Free, open-source, computational platform
- Public & user-friendly codes using HyTE
- Implemented in Matlab



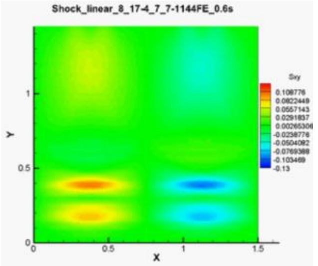
What is FreeHyTE for?

- Solving elliptic, parabolic and hyperbolic problems

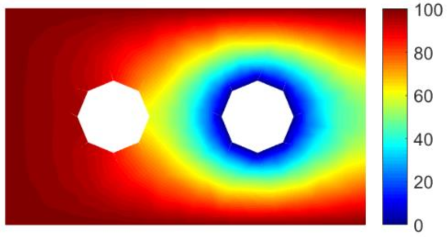
FreeHyTE: a hybrid-Trefftz finite element platform

Which modules are available at the moment?

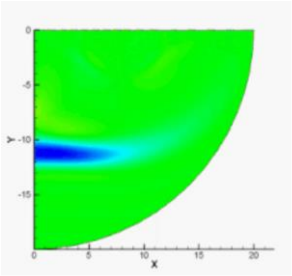
Solid transient



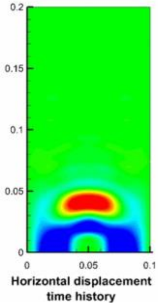
Direct Boundary Methods



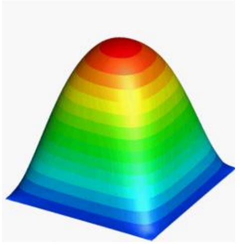
Triphasic transient



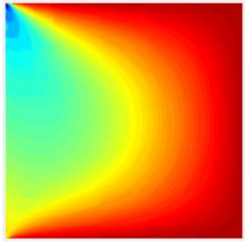
Biphasic transient



Transient heat



Elastostatic plane stress and strain structural problems

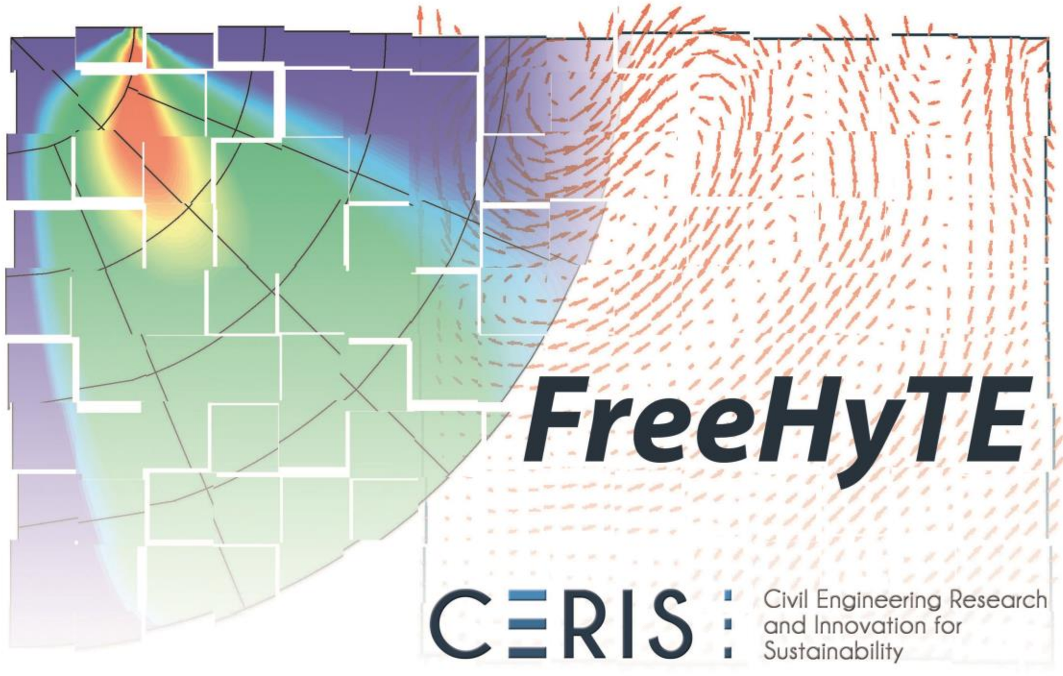


Which modules are available at the moment?

- Plane elasticity
- Porous media dynamics
- Transient problems
- Transient acoustics
- General Poisson, Laplace, and Helmholtz problems

FreeHyTE: a hybrid-Trefftz finite element platform

Why using FreeHyTE?



Why using FreeHyTE?

- Because HyTE just work well!
- Researchers don’t have to start from scratch
- Public & user-friendly codes using HyTE are rare (at best)
- Modular structure
- New modules are constantly added
- New versions of the existing modules are constantly released
- Easy to use

Application

FreeHyTE: a hybrid-Trefftz finite element platform

Example

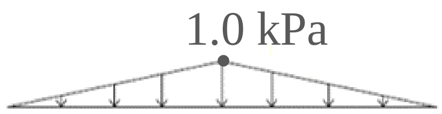
Problem

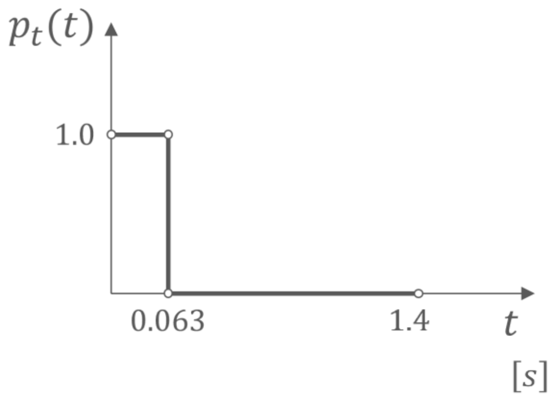
Solve the hyperbolic problem for the initial conditions:

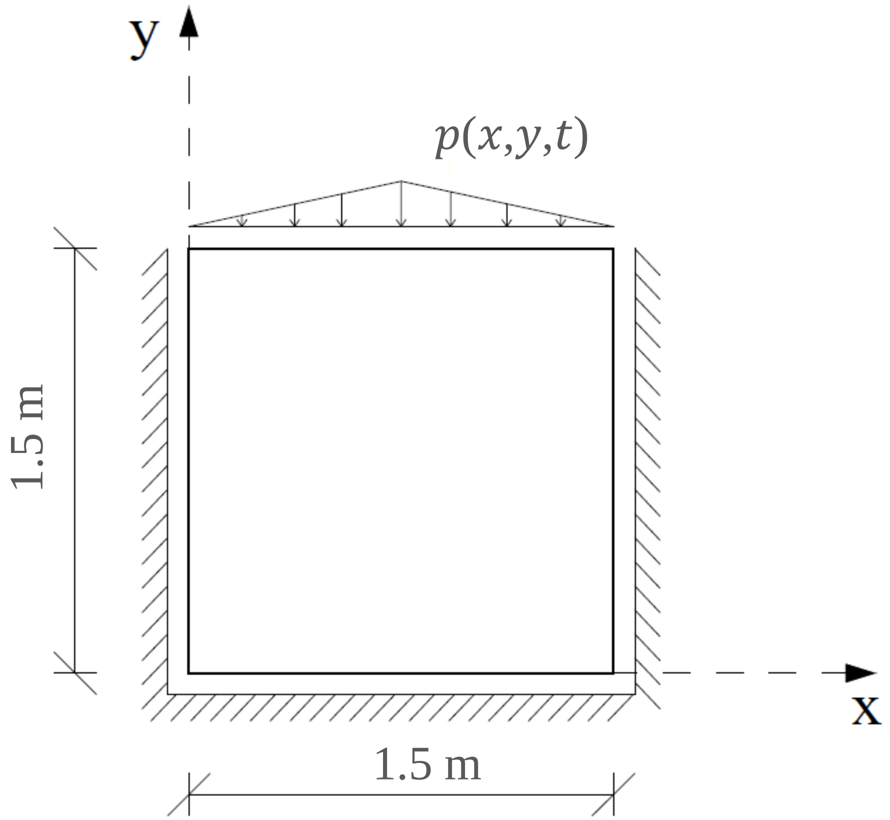
$$u_0(x,y) = 0$$
$$v_0(x,y) = 0$$
$$a_0(x,y) = 0$$

$$p(x,y,t) = p(x,y) \cdot p_t(t)$$

$$p\,(x, 1.5) = \begin{cases} \frac{2x}{b} & \text{for } x \leq \frac{b}{2} \\ -\frac{2x}{b} + 2 & \text{for } x > \frac{b}{2} \end{cases}$$



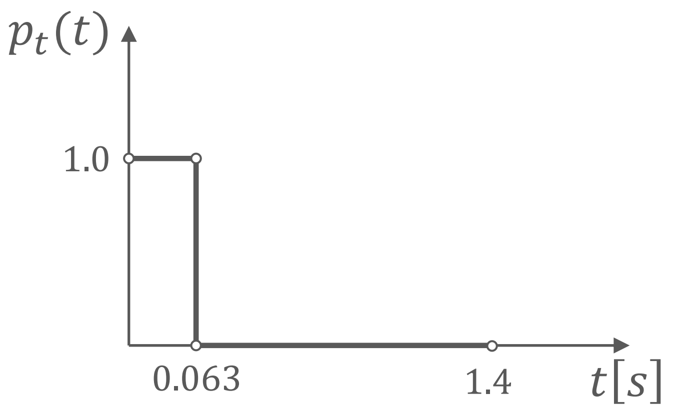




Discretization in time

Newmark method

- Total time: 1.4 s
- Time step: 0.015625 s



FreeHyTE: a hybrid-Trefftz finite element platform

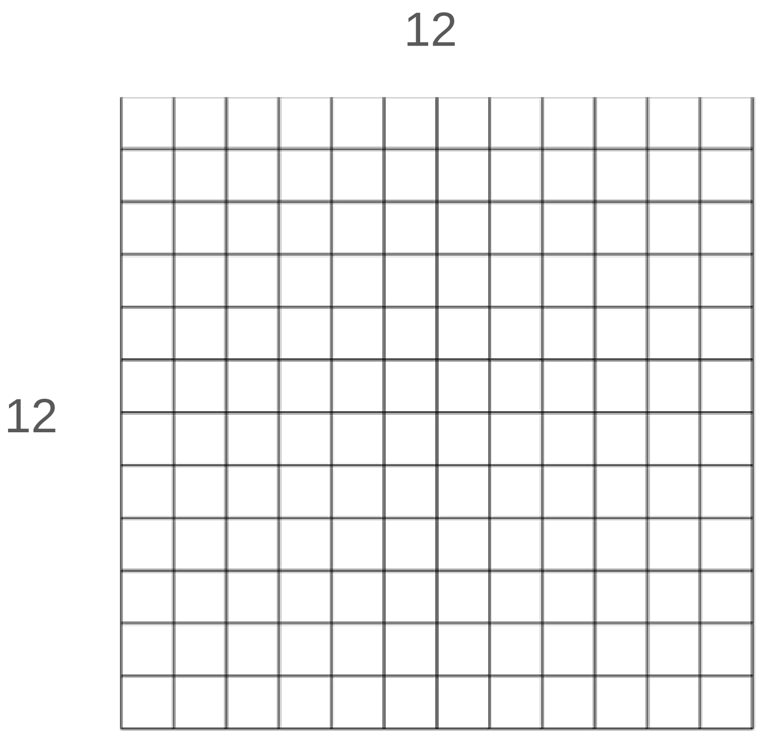
Example

Discretization in space

- **Complementary solution**:
    - 144 hybrid-Trefftz finite elements
    - Domain order: 17
    - Edges order: 8

- **Particular solution**:
    - Order: 17
    - Built on three pseudo-frequencies, $\lambda = \{1, 2, 3\}$

      $\lambda_1 = 0.1$; $\lambda_2 = 0.2$; $\lambda_3 = 0.3$



FreeHyTE: a hybrid-Trefftz finite element platform

Example

Results

Sy

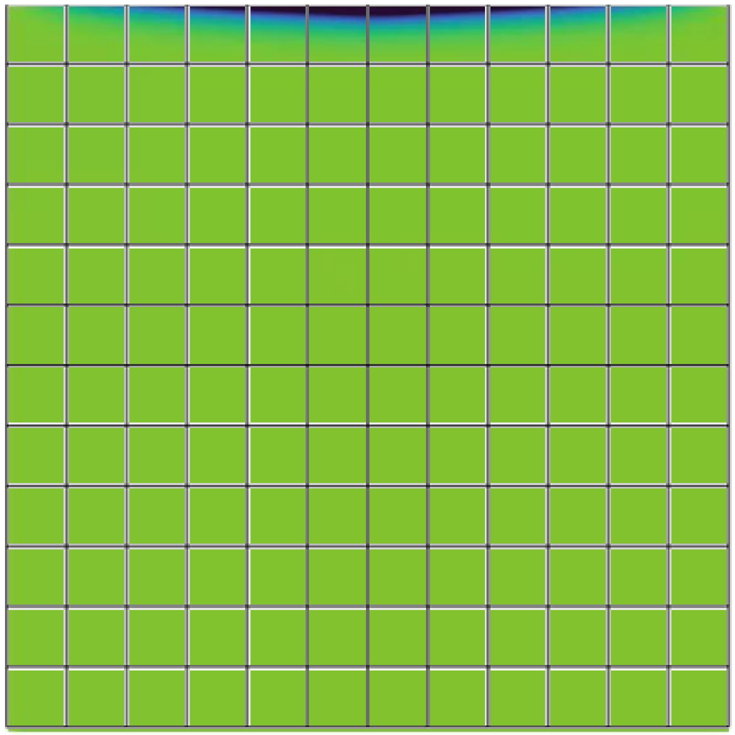
- Boundary conditions are well observed
- No visible field discontinuities

FreeHyTE: a hybrid-Trefftz finite element platform

Example

Results

Sy



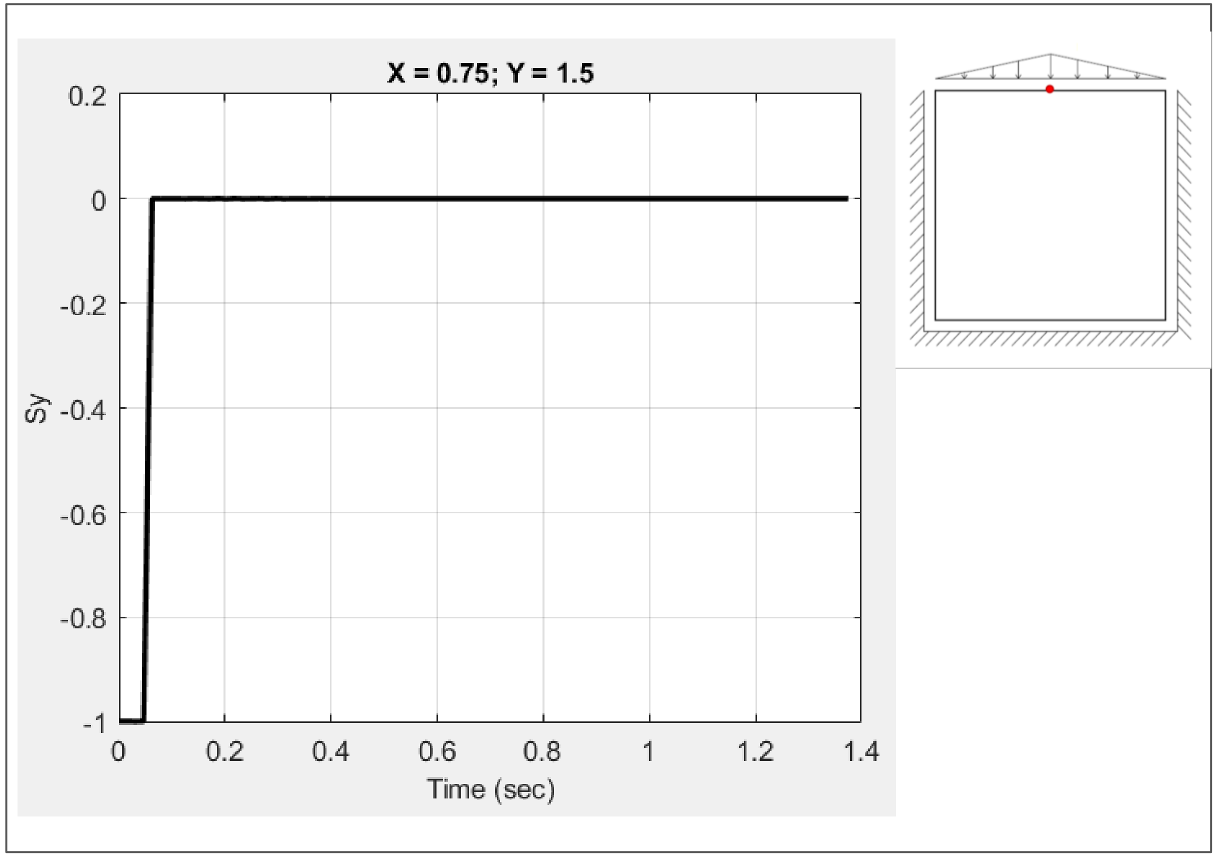
- Boundary conditions are well observed
- No visible field discontinuities

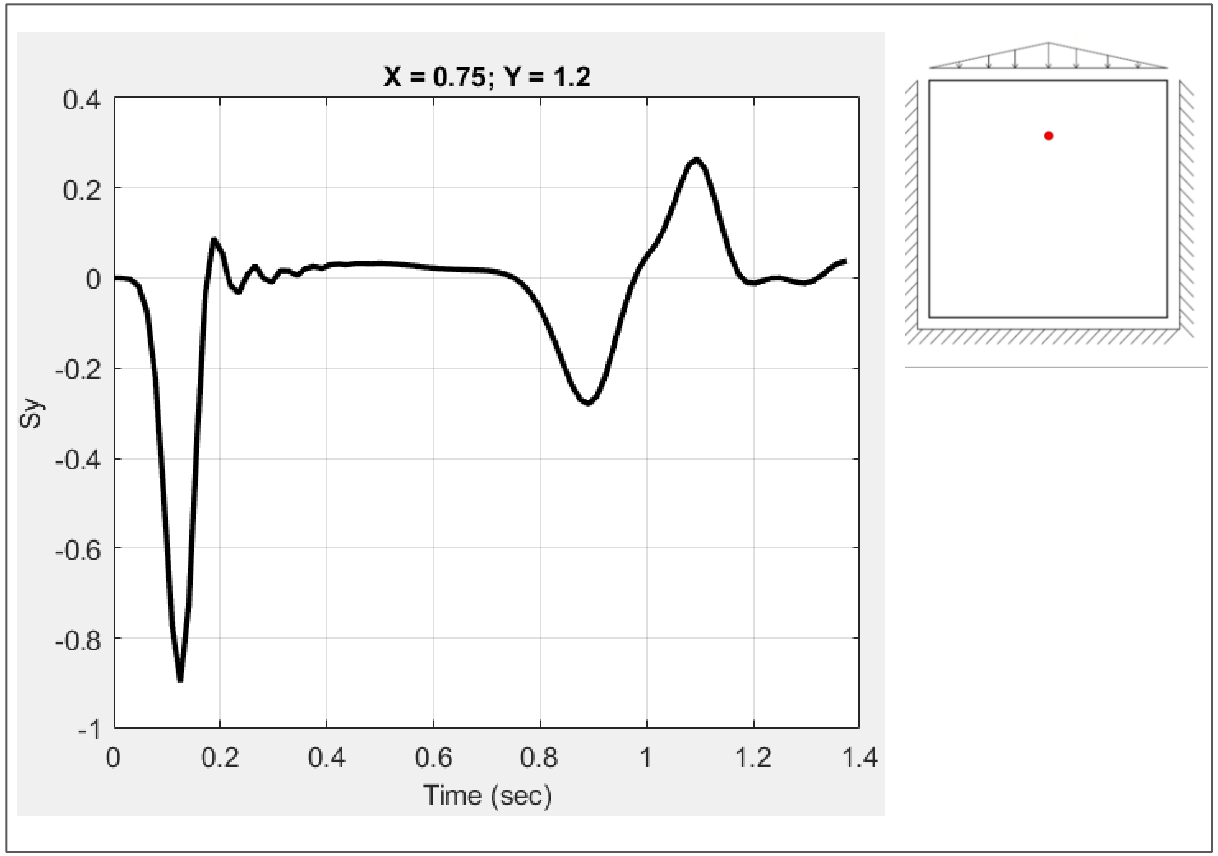
FreeHyTE: a hybrid-Trefftz finite element platform

Example

Results

Time history graphs





INTENT Project

Objectives

Intelligent health monitoring of road infrastructures using bender elements embedded in pavements (INTENT)

Objectives:

- ✓ develop finite element models for long term behavior of unbound granular layers subjected to cyclic loading
- ✓ develop supervised machine learning algorithms for damage identification in pavements

INTENT Project https://intent.ulusofona.pt/

FreeHyTE Project Open Source hybrid-Trefftz modules

https://sites.google.com/site/ionutdmoldovan/freehyte

Thank you for your attention!